Delft Wood Science Foundation Publication Series 2006, nr. ? - ISSN 1871-675X

Derivation of the shear strength of continuous beams.

As continuation on the theoretical explanation of the bearing strengths of locally loaded blocks [1], the braching model [2] is extended here and it is shown that, with the right dimensioning, always the shear strength is determining.
The elastic-plastic beam theory is extended for the influence of normal force and shear in [2]. Based on this extension the apparent contradictory test results of [4] of the shear- and bending strengths of beams and continuous beams is explained and also the shear and braching action of beams loaded close to the supports is derived and verified by tests of [3].
It appears that the theory of elasticity is not able to explain the data and to give the right stress distribution in two span beams, underestimating the bearing capacity by a factor 2/3, while the elastic-plastic beam theory according to Fig. 1 gives a very precise description of the data and the determining shear- and bending strengths.
The derivations, confirmed by tests, lead to requirements for design rules of the Codes.

dr. ir. T.A.C.M. van der Put

of the
Delft Wood Science Foundation
em. assoc. Research Professor of the
Technical University of Delft

DelftWoodScienceFoundation@xs4all.nl

Wielengahof 16
NL 2625 LJ Delft

Tel. +31152851980

Introduction

Because there is a renewed attention for the bearing and shear strength of beams with new proposals for the Eurocode based on empirical rules, it is necessary to reconsider the theory that always should be the basis of design rules and of testing. Therefore, after the discussion of the bearing strength given in [1], the derivation of the shear strength and the braching action of [2] will be discussed and extended here using the test results of [3] and [4]. This leads to the requirements for the design rules that should be satisfied. See the conclusions for the results.

Shear strength of beams

When there is plastic flow in compression, shear can only be carried in the elastic region. According to Fig. 1 is for bending of a beam of width b, loaded by a moment M and a shear force V:

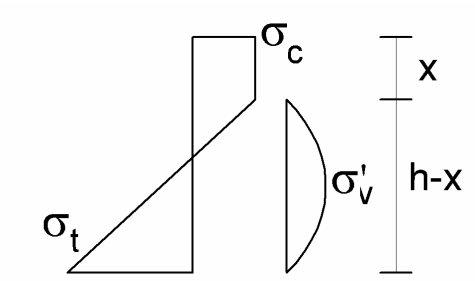

Fig. 1.- Bending and shear stresses .

$$\frac{M}{b} = \frac{\sigma_c + \sigma_t}{2}(h - x)\left(\frac{h}{2} - \frac{h - x}{3}\right) \tag{1}$$

The resultant normal force is zero, thus:

$$\sigma_c = \frac{\sigma_t + \sigma_c}{2}\left(1 - \frac{x}{h}\right) \tag{2}$$

Elimination of x/h from eq.(1) and (2) gives for σ_m, the quasi linear bending stress:

$$\sigma_m = \frac{6M}{bh^2} = \sigma_c \frac{3\sigma_t - \sigma_c}{\sigma_t + \sigma_c} \tag{3}$$

The total shear force V is:

$$V = \frac{2}{3}\sigma'_v bh\left(1 - \frac{x}{h}\right),$$

or by substitution of 1 - x/h from eq.(2):

$$\sigma_v = \frac{3V}{2bh} = \frac{2\sigma_c \sigma'_v}{\sigma_c + \sigma_t} \tag{4}$$

where σ_v is the quasi linear elastic shear stress divided over the total height "h".
At bending failure is: $\sigma_c = f_c$, or $\sigma_t = f_t$. At shear failure is $\sigma'_v = f'_v$.
For failure in bending and shear, there is a critical value of the shear slenderness $M_u / V_u h$ where the ultimate bending strength is reached at the same time as the ultimate shear stress. In the test of Fig. 2 is, according to eq.(3) and (4):

$$\frac{M_u}{V_u h} = \frac{a_c}{h} = \frac{3f_t - f_c}{8f'_v} = \frac{f_m}{4f_v} \approx 3 \tag{5}$$

The value 3 is mostly assumed for common dimensions and strength classes. This explains why the strengths f_m according to Fig. 2 (bending failure) and 3 (shear - bending failure) may be the same. The meaning of a/h is given in Fig. 2. Above the critical value of a/h, shear is not determining and there is bending failure with $\sigma_c = f_c$

and $\sigma_t = f_t$. Below this value, rotation and bending strength is reduced by the high shear force reducing the length x until x = 0 (at M/Vh ≈ 1 to 1.5, depending on the grade). Then the maximal possible shear strength is reached: $V_u = 0.67f'_v bh$ $= 0.67f_v bh$, at a moment: $M_u = f_m bh^2/6 = f_c bh^2/6$ ($f_t > f_c$ because f_t is the bending tensile stress that, by the volume effect, is about1.7 times the real tensile strength). With this beam theory, the test result of [4] can be explained. This was shown in a reaction on the preliminary publication of [4] that was send to the author, but not used in his final version of [4] and thus only is published in [2] of that meeting.

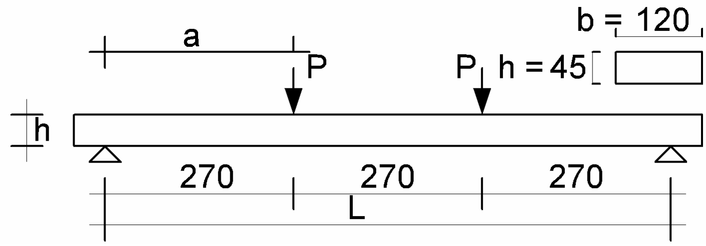

Fig. 2. – Test specimen for the bending strength, L/h = 18, sample size 50.
$f_m = 77.8$ MPa with $\sigma_v = 3.2$ MPa (< f_s, no shear failures).

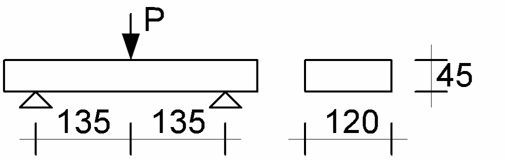

Fig. 3. – Test specimen for the shear strength, L/h = 6, sample size 70.
$\sigma_m = 64.8$ MPa and $f_v = 5.4$ MPa (only shear failures).

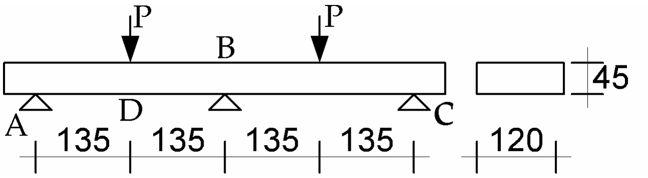

Fig. 4. – Australian test specimen for the shear strength, L/h = 6, sample size 14..
$f_m = 50.0$ MPa with $\sigma_v = 7.6$ MPa (2 specimens failed in shear).

In [4] the following dilemma was given for high quality wood, LVL (laminated veneer lumber):
- Fig. 3 and 4 show a lower bending strength than Fig. 2, although the reverse is expected by the volume effect.
- The shear stress of 7.6 MPa of Fig. 4 is too high above the strength of 5.4 MPa according to Fig. 3, while the bending strength is lower.

Before explaining this, first the elastic moment distribution of the beam on 3 supports, occurring at first flow, is determined. A cut of the beam at the middle support at point B (Fig. 4) will give a rotation at B by the loading P of: $\varphi = PL^2/16EI$. Only the non-symmetrical shear strain due to M_B/L will also give a rotation at B.

The moment at support B to close the gap gives a contrary rotation of $\varphi' = M_B L/3EI$. However the shear by the reaction M_B/L of this moment also closes the gap by: $\gamma = \tau/G \approx M_B/LbhG$. Thus:

$$\varphi - \gamma = \frac{PL^2}{16EI} - \frac{M_B}{LbhG} = \frac{M_B L}{3EI} \quad \text{or:} \quad M_B = \frac{3PL}{16} \cdot \frac{1}{1+4h^2/L^2} \tag{6}$$

With: h = 45; L =270 is: $M_B =$ 0.9·3PL/16. Thus $\sigma_m =$ 0.9.50 = 45 MPa.

It now appears that the field- and support moments are equal and also that $\sigma_m \approx f_c \approx 45\,\text{MPa}$.

The shear slenderness: M/Vh of the field moment at the side of the free support is: M/Vh = L/2h = 3 what is not determining. At the mid-support is $M_B/V_B h \approx L/4h \approx 1.5$ $(= f_m/4f_v = 45/(4\cdot 7.6) \approx 1.5)$.

In general is according to eq.(5) and (3), with $\alpha = \sigma_t/f_c$:

$$\frac{M}{Vh} = \frac{f_m}{4f_v} = \frac{3\alpha-1}{\alpha+1}\cdot\frac{f_c}{4f_v}, \quad \text{or at Point B:} \quad 1.5 = \frac{3\alpha-1}{\alpha+1}\cdot\frac{45}{4\cdot 7.6} \quad \text{giving:} \quad \alpha \approx 1,$$

showing that there is just no plastic flow and indicating that the maximal bending stress is: $\sigma_m = f_c$ = 45 MPa and the maximal shear stress is: $f_v = f'_v = 7.6$ MPa.

For Fig. 3 now applies: M/Vh = L/2h = 3 and f_v =5.4 MPa, or:

$$3 = \frac{3\alpha-1}{\alpha+1}\cdot\frac{45}{4\cdot 5.4} \quad \text{or} \quad \alpha = 1.56, \quad \text{giving a bending strength of:}$$

$\sigma_m =$ 45(3·1.56 – 1)/(1 + 1.56) = 64.9 MPa,

in agreement with the measured value of 64.8 MPa.

The bending strength of the bending test of Fig. 2 is: $f_m =$ 77.8 MPa. Thus:

$$77.8 = 45\cdot\frac{3\alpha-1}{\alpha+1} \quad \text{or} \quad \alpha = 2.15\,.$$

as is common for high quality wood [2]. The maximal shear stress of 7.6 occurs at the neutral line at point B. For shear failure at plastic flow in compression, as in Fig. 2, the maximal shear stress is combined with a tension stress and will be, also due to the volume effect, about 0.9 times lower. Thus: $f'_v = 0.9\cdot f_v = 0.9\cdot 7.6 = 6.8$ MPa. This means that the real shear strength at the maximal bending strength will be:

$$f_{v,m} = \frac{2f'_v}{\alpha+1} = \frac{2\cdot 6.8}{1+2.15} = 4.3\,\text{MPa}, \tag{7}$$

that will occur in the test at: a/h = (3·2.15 - 1)·45/(3.15·4·4.3) = 4.5.

Thus the 2-point bending test of Fig. 3 can be repeated with the load at a distance of 203 mm from the support to obtain the shear strength at ultimate bending without bending strength reduction.

In this comparison of different beams and loading cases it is assumed that corrections for volume effect in bending, as for clear wood, can be ignored for LVL. If there is any effect, it will be given by the values of α.

Because in Fig. 4 the field and support moments are equal and M_B is equal to the linear elastic ultimate moment due to the high shear loading, this also should be the case for the field moment and a brittle failure in bending of both moments at the same time is to be expected. This is not reported in [4] and from the tests of [3] it follows that by the high shear stress, there is stress redistribution and a flow in shear, making the gap between the beams AB and CB, to be closed by M_B, much smaller, reducing M_B and providing compatibility for flow of the field bending moments.

Shear strength of close to the support loaded two span beams

In [3], two series of tests were done for concrete formworks according to Fig. 5 and 6 with variable values of a. Here the calculated $M_B / V_B h$ values range from 0.9 to 2.6, giving an extension around the value of 1.5 of Fig. 4.

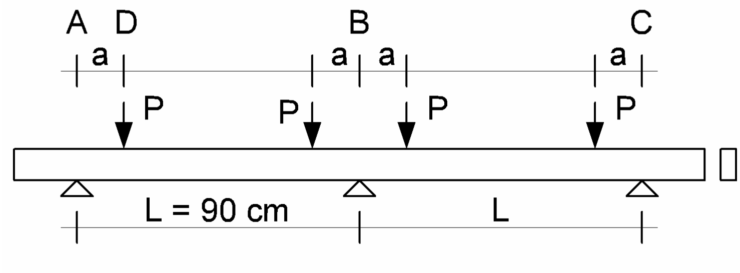

Fig. 5. – Series A, a = 8, 12, 16 and 24 cm; L/h = 11.6, sample size 4x5 = 20
bxh = 59x78 mm².

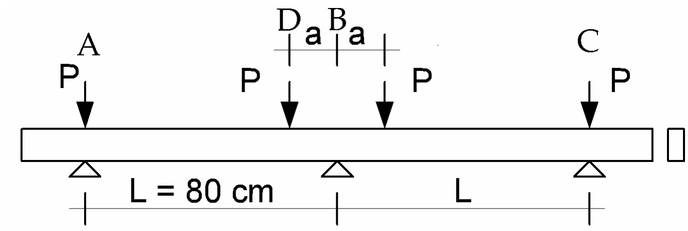

Fig. 6 – Series B, a = 8, 12, 14, 16, 24 cm; L/h = 10.3, sample size 5x4 = 20
bxh = 59x78 mm².

The results of the tests are given in Table 1 and 2. The stresses follow from:

$\sigma_{mB} = 6M_B / bh^2; \sigma_v = 1.5V / bh; \sigma_{c,90} = R_B / A_s$

For series A, the reactions R, shear forces V and moments M are:

$R_A = R_C = P(1 - 3a/2L + 3a^2/2L^2) = V_A$

$R_B = 2P(1 + 3a/2L - 3a^2/2L^2) = 2V_B$

$M_B = -1.5Pa(1 - a/L)$

$M_P = M_D = Pa(1 - 3a/2L + 3a^2/2L^2)$

The failure modes of Series A are as follows:

At a = 8 cm, failure occurs by compression perpendicular to the grain and secondary failure after a huge deformation (flow) at the loading points.

At a = 12 cm, there also is a strong deformation at the loading points. Failure mostly occurs by bending in the field at knots.

At a = 16 cm, bending failure occurs in the field at knots.

At a = 24 cm, all beams failed by bending in the field and 2 beams also by failures at the middle support.

Table 1: Series A, stresses at failure in MPa

a cm	beam nr.	$\sigma_{c,90}$	σ_v	$\sigma_{m,B}$	$\sigma_{m,P}$	grade	
	3a	8.6	8.2	41.2	26.6		Failure by compression
	3b	9.8	9.6	49.2	31.7		Shear-bending failure
8	6b	8.6	8.5	43.0	27.7	+	Failure by compression
	8b	7.4	7.1	35.6	22.9		"
	17a	6.6	6.4	32.6	21.0		"
	5b	6.7	6.5	43.9	27.8		One sided failure
	11b	8.8	8.8	62.0	39.4		Shear failure
12	15b	6.6	6.3	42.3	26.8	+	Bending failure at knots
	17b	6.7	6.4	43.5	27.6		"
	18b	6.6	6.3	43.0	27.3	-	"
	5a	4.8	4.6	38.4	24.3	-	Failure at P by knots
	6a	7.2	6.9	57.0	36.0	+	Also failure at point B
16	10b	6.9	6.6	54.9	34.8	+	Failure at P by knots
	11a	8.3	7.9	65.0	41.1		Shear failure
	12b	5.9	5.6	45.8	29.0	-	Failure at P by knots
	8a	5.4	5.3	56.0	35.9	-	Knots in the failure zone
	10a	6.0	5.8	60.8	39.0	+	"
24	12a	5.3	5.2	55.6	35.6	-	"
	15a	5.2	5.0	53.0	34.0	+	bending failure
	18a	4.9	4.7	49.0	31.4		Knots in the failure zone

The lower of the 2 Dutch strength classes at that time was applied, however, + means wood of the highest class, and – means below the lowest class Beams of 2.3 m were sawn in the length, giving 2 paired beams "a" and "b".
Moisture content: 20% (18 to 22%).
Underlined values means: determining location and values for failure.

For series B, the reactions R, shear forces V and moments M are:

$$R_A = R_B = P + P(a^2/L^2)(1.5 - a/2L) = V_A$$

$$R_B = 2P(1 - a/L)(1 + a/L - a^2/2L^2) = 2V_B$$

$$M_B = -Pa(1 - a/L)(1 - a/2L)$$

$$M_P = M_D = P(a^2/2L)(3 - a/L)(1 - a/L)$$

$$\sigma_{mB} = 6M_B/bh^2; \sigma_v = 1.5V/bh; \sigma_{c,90} = R_B/A_s;\ A_s = 5.87\text{x}10 \text{ cm}^2.$$

The failure modes of Series B are as follows:

At a = 8 cm, failure occurred by compression, pressing the wood fully together. Secondary failure occurred in 2 beams after strong deformation and cut through of the fibers by the steel plates of the middle support.
At a = 12 cm, there also is a strong deformation at the support and loading points. Failure occurred by shear and in one case secondary bending failure occurred after shear failure.
At a = 14 cm, bending failure occurred at the middle support
At a = 16 cm, bending failure occurred at the middle support and a start of shear failure in 2 beams. failure started in shear
At a = 24 cm, all beams failed by bending at the middle support.

Table 2: Series B, stresses at failure in MPa

a cm	beam nr.	$\sigma_{c,90}$	σ_v	$\sigma_{m,B}$	$\sigma_{m,P}$	grade	
	1a	<u>6.4</u>	6.3	<u>22.5</u>	3.4		Cut through at B
	7a	<u>7.5</u>	7.4	25.3	3.8		Failure by compression
8	14a	<u>8.0</u>	7.8	27.8	4.3		"
	19a	<u>5.9</u>	5.9	<u>20.9</u>	3.2	-	Cut through at B
	9b	7.0	<u>7.0</u>	35.4	8.2		Shear failure
	14b	7.0	<u>6.9</u>	34.6	8.0		"
12	16a	7.6	<u>7.5</u>	<u>37.2</u>	8.6	-	Shear + secondary bend.
	21a	7.9	<u>7.9</u>	39.4	9.1		Shear failure
	1b	6.0	5.8	<u>32.4</u>	9.3		Bending failure
	9a	7.2	<u>6.9</u>	<u>38.5</u>	11.0		Shear + bending
14	13b	7.9	<u>7.8</u>	42.0	12.0	-	Shear + start bending
	20a	7.0	6.8	<u>37.8</u>	10.8		2 bending failures at B
	4a	7.6	<u>7.5</u>	<u>48.0</u>	15.0		Bending + shear
	16b	7.2	<u>7.1</u>	<u>43.4</u>	14.2		"
16	19b	6.6	6.5	<u>40.8</u>	12.5		Bending failure at B
	20b	5.7	5.7	<u>36.2</u>	11.3	-	"
	4b	6.6	6.4	<u>54.8</u>	<u>25.7</u>		Bending failures at B + P
	7b	5.5	5.3	<u>44.8</u>	21.5		Bending failure at B
24	13a	5.6	5.2	<u>43.2</u>	20.6	-	"
	21b	6.5	6.2	<u>52.0</u>	24.8		"

<u>Discussion of the test results</u>
In Table 1 of Series A, at a = 24 cm, all beams failed by bending in the field although the bending moment at support B is 1.6 times higher. This can not be explained by a volume effect or a round off of the moment-peaks by the fact that the reaction is not a

point load, because then the strength should also strongly increase with smaller values of a/h and the reverse is occurring. According to [5], the volume effect for bending is: $f_m / f_{m,0} = c \cdot (200/h)^{0.11}$, where "c" ranges from c = 1.05 when L/h = 35 to c = 1.15 when L/h = 7. Thus $f_m / f_{m,0} = (L_0 / L)^{0.0565} \cdot (200/h)^{0.11}$. For equal heights, the determining strength ratio by the volume effect thus is:
$M_B / M_P = ((L - a/2)/(a/2))^{0.0565} = ((80-12)/(12))^{0.0565} = 1.1$, while the round off effect of M_B is of the same order: 0.9, showing the total influence of these effects to be negligible and there thus should be a strong moment redistribution by plasticity. Flow in compression perpendicular to the grain in the oblique shore direction, causes also flow in shear deformation of the beam cross section at B, reducing strongly M_B, the moment at the support. This shear deformation at the B cross section can be seen on the photo for a/h = 1 and this deformation needs not to be symmetrical and for a higher value of a/h =16, even a pure shear mechanism did occur (see photo). There thus clearly is a moment redistribution reducing M_B at the end to be equal to the field moment as shown by the failure of both moments in e.g. beam 8a and 10a. The calculation of the real failure stresses thus should not be based on linear elasticity but on a mechanism, according to the theory of plasticity as will be discussed later.

Table 3: Mean values of the strengths of Table 1 and 2 in MPa.

a cm	$\sigma_{c,90}$	σ_v	$\sigma_{m,B}$	$\sigma_{m,P}$	
8	8.2	8.0	40.3	26.0	Series A
12	7.1	6.9	46.9	29.8	4x5 specimens
16	6.6	6.3	52.2	33.0	
24	5.4	5.2	54.9	35.2	
8	7.0	6.9	24.1	3.7	Series B
12	7.4	7.3	36.7	8.5	5x4 specimens
14	7.0	6.8	37.7	10.8	
16	6.8	6.7	42.1	13.2	
24	6.0	5.8	48.7	23.2	

Outer determining bending failures at a = 24 cm, also failure by compression perpendicular to the grain was determining at a = 8 cm while failure by shear combined with compression and bending occurred in all other cases.

Explanation of the measured shear strength.

The shear strength of a large number of tests can e.g. be found in [5] and the regression line of all tests of shear in bending, shear in torsion and block shear is:

$f_v = 20.95 - 3.35 \log A_v$, (8)

where f_v is in MPa and A_v is the sheared area in mm^2.

Omitting the block tests, the regression line is:

$f_v = 19.20 - 3.03 \log A_v$ (9)

For the tests of [3], the values of A_v = b x a are: for b = 58.7 mm and a = 80, 120, 140, 160 and 240 mm are given in table 4. For the median value of A_v = 58.7 x 140 = 8218 cm², the reference value is:

$f_{v,0} = 20.95 - 3.35 \log 8218 = 7.8 \text{ MPa}$, (10)

and eq.(8) can be written:

$$f_v - f_{v,0} = -3.35 \log(A_v / 8218) = -3.35 \cdot 0.434 \ln(A_v / 8218) = -1.455 \ln(A_v / 8218),$$

$$f_v / f_{v,0} = 1 - 0.186 \cdot \ln(A_v / 8218)$$

According to the theory of the Appendix is:

$n = [\partial(f_v / f_{v,0}) / \partial(A_v / 8218)]_{A=8218} = -0.186$, and the last equation becomes:

$$f_v / f_{v,0} = \left(A_v / 8218\right)^{-0.186}$$

The other regression line eq.(9) gives n = 0.18 and $f_{v,0}$ = 7.3 MPa. For the data of [3], $f_{v,0}$ is still lower, $f_{v,0}$ = 6.8 MPa, probably because only bending with shear is involved and due to the higher moisture content and lower grade.

The power law representation of the regression line gives a meaning to the data to represent the volume effect according to the Weibull weakest link theory. The variation coefficient for the occurring, for failure determining disturbances, is 1.2·0.186 = 0.22.

Because f_v is not very sensitive for the value of "n", a rounded value of n = 0.2 can be chosen, the same value as in the Eurocode for larger dimensions, leading to:

$$f_v = f_{v,0} \cdot \left(\frac{A_v}{8218}\right)^{-0.2}, \qquad (11)$$

giving a precise fit in Table 4 and an explanation of the strength to be determined by the shear strength in probably all cases.

Table 4: Theoretical extrapolated first flow values of f_v in MPa.

a cm	$A_v = b \cdot a$ mm^2	Theory Eq.(11) f_v	Measure-ments Series A+B f_v
8	4696	7.6	7.5
12	7044	7.0	7.1
14	8218	6.8	6.8
16	9392	6.6	6.5
24	14088	6.1	(5.5) ?

Determination of the bending strength.

For a pure shear flow mechanism over some length of the beam, the opposite shear forces have to be equal and also the end moments. Thus for series B, the field moment M_P is equal to M_B, the moment at the support as given in Fig. 7. For series A, this also is the case because of the always occurring bending flow mechanisms. According to Fig. 7 is, for equal moments is: $M_D - M_B(L-a)/L = M_B$ or: $Pa(1\text{-}a/L) - M_B(L-a)/L = M_B$ or:

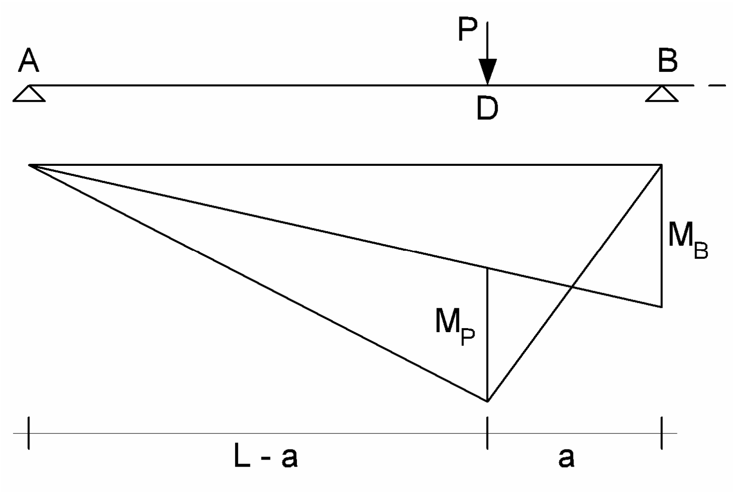

Fig. 7: Equal field and support moments $M_P = M_B$ of Series B.

Table 5: Measured strength values at (shear-) flow.

a cm	M/Vh	P kN	$\sigma_{m,B}$ $\sigma_{m,P}$ MPa	σ_v MPa	$\sigma_{c,90}$ MPa	
8	0.9	21.45	26.3	7.6	7.9	Series A
12	1.2	17.70	31.3	6.5	6.7	
16	1.5	15.95	36.2	6.0	6.2	
24	2.0	12.20	38.6	4.8	5.0	
8	0.5	21.15	13.4	6.5	6.8	Series B
12	0.8	22.85	21.1	6.8	7.1	
14	0.9	21.55	22.8	6.35	6.6	
16	1.0	21.45	25.5	6.2	6.5	
24	1.5	20.00	33.0	5.4	5.6	

$$M_B = Pa\frac{1-a/L}{2-a/L} \tag{12}$$

$$V_B = P\frac{L-a}{L} + \frac{M_B}{L} = P\frac{1-a/L}{1-a/2L} \tag{13}$$

$$\frac{M_B}{V_B h} = \frac{a}{2h} \tag{14}$$

The same applies for Series A, leading for the highest shear force to:

$$M_B = \frac{Pa}{1 + a/L}, \qquad V_B = P\frac{1 + 2a/L}{1 + a/L}, \qquad \frac{M_B}{V_B h} = \frac{a}{h} \cdot \frac{1}{1 + 2a/L} \tag{15}$$

These equations result to the strength values given in Table 5.

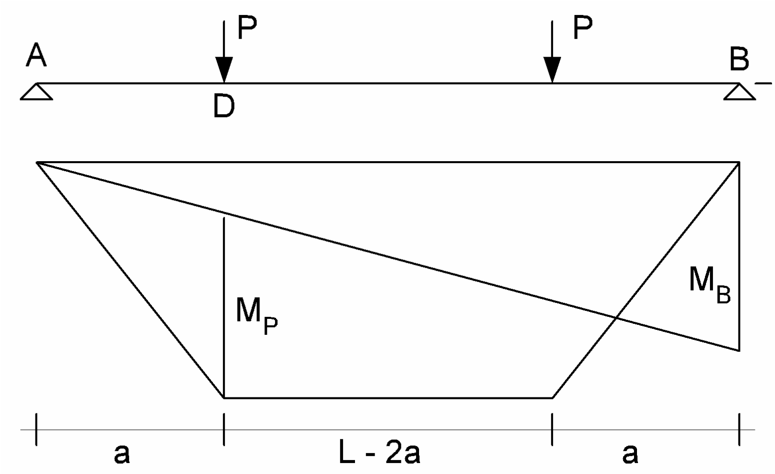

Fig. 8: Equal field and support moments $M_P = M_B$ of Series A.

The shear strength according to eq.(11), adapted to the strength of 6.35 MPa at a = 14 cm, becomes:

$$f_v = 6.35 \cdot \left(\frac{A_v}{8218} \right)^{-0.2} \tag{16}$$

and is given in Table 6.

Table 6: Theoretical flow values of f_v in MPa.

a cm	$A_v = b \cdot a$ mm^2	Theory Eq.(16) f_v	Measure-ments Series A f_v	Measure-ments Series B f_v	Measure-ments mean A +B f_v
8	4696	7.1	7.6	(6.5)	7.1
12	7044	6.6	6.5	6.8	6.6
14	8218	6.4		6.4	6.4
16	9392	6.2	6.0	6.2	6.1
24	14088	5.7	4.8	5.4	5.1

The data of f_v suggest the same cause and type of shear failure in series A and B what is shown in Table 4 and Table 6 by calculating the mean value of the shear strengths of both series that appears to follow the theoretical eq.(16) precisely.

However this does not explain the same steep increase of the bending strength f_m in both series as function of M/Vh between M/Vh = 0.9 and 1.5, that occurs at different values of "a" in series A and B. The explanation of this behaviour follows from the beam theory (see Fig. 1 and equations (1) to (5)) that shows a decrease of the ultimate shear force with the increase of M/Vh and an increase of the bending rotation and thus an increase of the bending moment. Series B, at a = 24 cm, did show only bending failure while for lower values of "a", combined bending and shear failures occurred. The boundary of this combined failure thus lies at a = 16 cm. This means for series B of Table 5 at a = 16 cm, that $\alpha = 1$, and $f_m = f_c = 25.5$ MPa and $f_v = f'_v =$ 6.2 MPa. For a = 24 cm then the ultimate combined shear-bending strength according to eq.(3) is: $f_m = f_c \cdot (3\alpha - 1)/(\alpha + 1) = 33 = 25.5 \cdot (3\alpha - 1)/(\alpha + 1)$, or: $\alpha = 1.34$. Thus $f_v = 2f'_v/(\alpha + 1) =$ 2·6.2/2.34 = 5.3 MPa, what agrees with the measured value of 5.4 MPa in Table 5.

The value of $\alpha = 1.34 = f_t / f_c = 1.7 \cdot f_{t,0} / f_c$, gives $f_{t,0} / f_c = 1.34/1.7 = 0.8$. Thus the tensile strength $f_{t,0}$ is 0.8 times the compression strength. The factor 1.7 is due to the volume effect of the bending tensile strength f_t with respect to the pure tensile strength $f_{t,0}$. Below M/Vh = 1 there is no flow in bending $\sigma_m < f_m$; $\sigma_{m,t} = \sigma_{m,c}$, or $\alpha = 1$. Thus the point where $f_m = f_c$ and $f_v = f'_v$ and $\alpha = 1$, occurs at M/Vh = 1 in series B. For series A, this point is found by interpolation in Table 5 between a = 8 and 12, where M/Vh = 0.9 to 1.2, giving $f_m = f_c =$ 28 MPa. With this value of f_c, the values of f_m / f_c for other values of "a" can be calculated and according to eq.(17), that is based on eq.(3), the values of α are known and are given in Table 7.

$$\alpha = (1 + f_m / f_c)/(3 - f_m / f_c) \qquad (17)$$

An adaptation of $f'_{v,0}$ of eq.(16), to give the value of 7.6 MPa at a = 8 cm, is:

$$f'_v = 6.8 \cdot (A_v / 8218)^{-0.2} \qquad (18)$$

This adaptation of f_v for the stronger series A is, according to eq.(4) or eq.(7):

$$f_v = 2f'_v/(\alpha + 1) \qquad (19)$$

Table 7: Theoretical strength values of series A, with f_v in MPa.

a cm	$A_v = b \cdot a$ mm^2	Theory Eq.(18) f'_v MPa	M/Vh	$f_{m,B}$ $f_{m,P}$ MPa	Theory α Eq.(17)	Theory Eq.(19) f_v MPa	Measure-ments f_v MPa
8	4696	7.6	0.9	(26.3)	~ 1	7.6	7.6
12	7044	7.0	1.2	31.3	1.13	6.6	6.5
16	9392	6.6	1.5	36.2	1.34	5.7	6.0
24	14088	6.1	2.0	38.6	1.47	4.9	4.8

Based on the database the bending strength at 20% m.c. is 35.4 MPa for un-graded wood at commercial sizes, thus for beams of at least twice the height of the test-specimens. Thus including the volume effect, the bending strength here is: $(2)^{0.11} \cdot 35.4 = 1.08 \cdot 35.4 = 38.2$ MPa. According to Table 7, the maximal bending strength thus is reached at a = 24 cm, at M/Vh = 2.

Braching model

As mentioned in Table 1 and 2, at a = 8 cm, failure by compression perpendicular to the grain, that shows no volume effect, starts to be determining for the strength at the chosen dimensions of the bearing plate at the support and there is a cut off of the by the volume effect increasing shearing strength for smaller values of a < 8 cm. This cut off also applies for bearing by one or two dowel joints. It is shown for many cases, the last time in [6], that the spreading model also applies for a load on a beam by a dowel, as given in Fig.9, leading to the embedding strength of:

$$f_s = 1.1 \cdot f'_{c,90} \cdot \sqrt{3a/d} \qquad (16)$$

On the spreading action, where the boundary stress σ is in equilibrium with the total force P on the bolt: $\sigma \cdot 2a \cdot b = P$, an equally but opposite stress system is superposed to give a stress free lower beam boundary. These superposed stresses are not determining outside the neighbourhood of the bolt, explaining why the model can be applied to the bearing strength of the bolt in a beam with free boundaries as in Fig. 9.

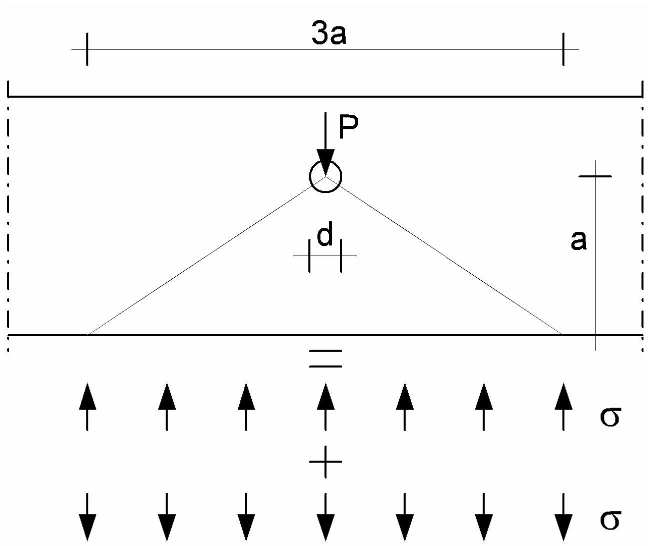

Fig. 9: Spreading model

As shown before and mentioned in [2], the shear strength, which depends on the volume effect, normally is determining for the strength of close to the support loaded beams, but the bearing capacity may be reduced further when the compression strength perpendicular to the grain is made determining. The compression strength of the inclined shore follows from the bearing strength, discussed in [1].

This bearing strength is:

$$k_c = \frac{\sigma_{s,\phi}}{\sigma_{c,\phi}} = c\sqrt{0.5 + \frac{3H/\cos\phi + L/\cos\phi}{2s/\cos\phi}} = 1.1\cdot\sqrt{0.5 + \frac{3H+L}{2s}} = \frac{f_s}{f_{c,90}} \tag{17}$$

thus is the same as for a not inclined shore.
The equality: $\sigma_{s,\phi}/\sigma_{c,\phi} = f_s/f_{c,90}$ follows from the maximum stress criterion perpendicular to the grain, that is a save lower bound of the strength for not too high angles ϕ because it does not contain the influence of hardening.
With L = 6 cm and s = 5 cm of the loading plates and H = 8 cm, is:

$$k_c = 1.1\cdot\sqrt{0.5 + \frac{3\cdot 8 + 6}{2\cdot 5}} = 2.1 \tag{18}$$

Thus $f_{c,90} = 7.3/2.1 = 3.5$ MPa as mean value of Table 5 of series A and B, at the determining value of a = 8 cm. This is comparable with the $f_{c,90}$ values of [1].

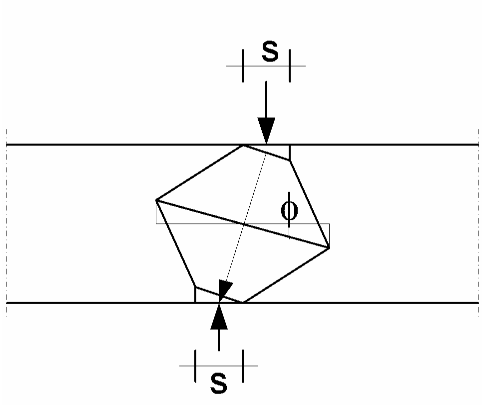

Fig. 10. – Bearing or braching mechanism,

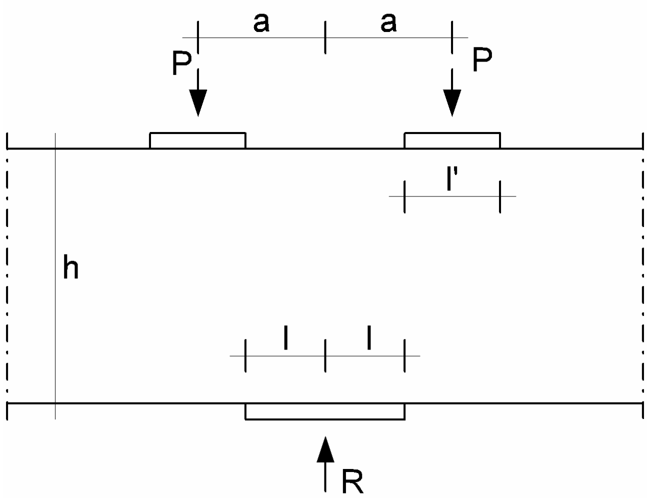

Fig. 11. – Just no overlap of bearing plates at a = 8 cm.

Because in this investigation the $\sigma_{c,90}$ values go down when "a" goes up from a = 8 to a = 24 cm, exactly the same way as the shear strength f_v, the shear strength is

determining and not the compression strength perpendicular to the grain of the braching action. This should be the case up to the situation of Fig. 11, because for smaller values of a < 8 cm, not the whole load R/2 is transmitted by shear. Thus half the length of the central bearing plate should be: $l = .67 \cdot f_v \cdot h / f_s = .67 \cdot 7.0 \cdot 7 \cdot 8/7.3 = 5$ cm, as is applied. Here $f_v = 7$ MPa is the shear strength at a = 8 cm, the mean value at a = 8 cm in Table 6. In general for the bearing length applies: l = 0.64·h ≈ (2/3)h.
Fr a middle support two times this value applies, thus

$$l = 1.27 \cdot h. \tag{19}$$

Because the spreading for combined shear failure is not higher than for compression failure in this case, this rule also should apply for design values of the strengths.

Biaxial failure criterion

The design rules for bearing blocks, [1], are based on flow and hardening in triaxial conditions by confined dilatation. This confinement often depends on the friction between wood and the steel bearing plate and not on structural means and for safety these rules are not used for combined stresses at the supports and loading points of beams because the Code rules for the strength under combined stresses and the Code failure criterion are based on test results of biaxial and uniaxial tests. For combined stresses in beams, therefore the real failure criterion of [7] has to be applied. It follows from [7] that the bending compression strength along the grain increases by compression perpendicular up to a maximum and then decreases at further increase of the compression perpendicular. For this reason, there still is no decrease of the bending compression strength when the compression perpendicular is about half of the uniaxial compression strength $f_{c,90}/2$ (in the weakest plane).

As long the exact approach of [7] is not followed, the compression stress perpendicular to the grain at a middle support should safely be limited to $f_{c,90}/2$ in order to maintain the ultimate compression stress of the bending strength of the beam. For end-supports, $f_{c,90}$ can be applied, as is common practice. Because of the spreading of the compression stresses perpendicular to the grain, the stress combination with shear is not determining in this case.

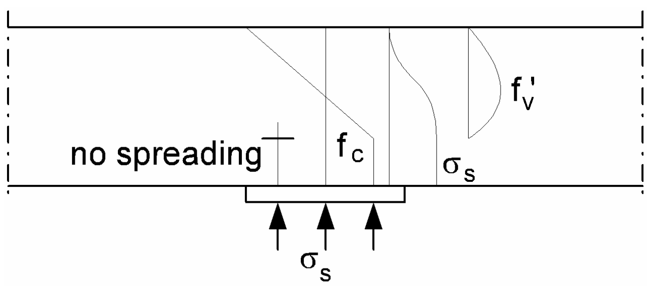

Fig. 12 – No spreading of σ_s due to bending flow stress f_c, limiting σ_s to $f_{c,90}/2$.

Conclusions

- The elastic-plastic beam theory is extended for the influence of shear force (Fig.1).
- Based on this extended beam theory, the apparent contradictory test results of [4] of the shear- and bending strengths of beams and continuous beams are precisely explained.
- It appears that the theory of elasticity is not able to explain the strength data of [3] and [4] and to give the right stress distribution in two span beams, underestimating the bearing capacity of the tested beams of [3] by a factor 2/3. The bending failure of series A occurred in the field although the bending moment at the middle support is 1.6 times higher than the field moment according to the theory of elasticity. There thus is a strong moment redistribution by plastic flow, that appears to be shear flow at the middle support.
- Flow in compression perpendicular to the grain in the oblique shore direction, causes also flow in shear deformation at the middle support reducing strongly the moment at this support and causing failure to start in the field although according to the theory of elasticity the field moment is a factor 1.6 lower than the moment at the middle support (see Table 1).
- The shear strength can be explained from the regression line of many tests of shear in bending, shear in torsion and block shear. This regression line can be transformed to a power law form representing the volume effect according to the Weibull weakest link theory giving a precise fit of the data of [3] showing the strength to be determined by the shear strength in all cases.
- According to the extended beam theory, there is a critical value of the shear slenderness M/Vh (the relative moment-shear force ratio) where the maximal ultimate moment is reached and at the same time as the maximal shear force is determining. Above this critical value, bending alone is determining by the same maximal ultimate moment. Below this critical value, the rotation and thus also the ultimate moment is reduced by the shear force that also is determining for failure (see Fig. 1).
- This critical value is about M/Vh = 3 to 4.5, depending on the wood quality as follows from the data of [4]. The data of [3] suggest even a possible lower critical value of M/Vh = 2 for lower grades.
- For M/Vh ≤ 1 there is a linear stress distribution along the height of the beam and no flow in bending $\sigma_m < f_m$; $\sigma_{m,t} = \sigma_{m,c}$, or $\alpha = 1$. The point where $f_m = f_c$ and $f_v = f'_v$ and $\alpha = 1$, occurs at M/Vh = 1 in the series of [3] and is 1.5 for the high quality laminated veneer lumber.
- As continuation on the theoretical explanation of the bearing strengths of locally loaded blocks [1], the braching model of beams loaded close to the supports [2] is extended and verified by tests of [3].
- Because the shear strength should be determining and not the compression strength

perpendicular to the grain of the braching action the cut off of the shear strength should not be earlier than in the situation of Fig. 11, when just the whole load is transmitted by shear. Therefore the length of the bearing plate should be : $l = 0.64 \cdot h$ and the length of the central bearing plate: $l = 1.27 \cdot h$.
This prescription is a simple rule for the Codes.
- For combined stresses at the supports and loading points of beams the failure criterion, used in the Codes, is based on biaxial and uniaxial tests. It follows from [7] that there then is no decrease of the bending compression strength when the compression perpendicular is about half of the uniaxial compression strength $f_{c,90}/2$ (in the weakest plane). As rule for the Codes, the compression stress perpendicular to the grain at a middle support should safely be limited to $f_{c,90}/2$ in order to maintain the ultimate compression stress of the bending strength of the beam. For end-supports, $f_{c,90}$ can be applied, as is already common practice.

Literature

[1] T.A.C.M. van der Put, Derivation of the bearing strength perpendicular to the grain of locally loaded timber blocks. Delft Wood Science Foundation Publication Series 2006 nr. 3, www.dwsf.nl at publications-downloads.
[2] T.A.C.M. van der Put, Discussion of the failure criterion for combined bending and compression, CIB-W18/24-6-1, Oxford, September 1991.
[3] P. Vermeijden, Shearing strength of close to the support loaded beams on 3 supports (in Dutch), Report 4-68-13/HE-2, TU-Delft, Sept. 1968
[4] R.H. Leicester and E.G. Young, Shear strength of continuous beams, CIB-W18/24-10-1, Oxford, September 1991
[5] H.J. Larsen, The Design of Timber Beams, CIB-W18/5-10-1, Karlsruhe, October 1975.
[6] T.A.C.M. van der Put and A.J.M. Leijten, Evaluation of perpendicular to the grain failure of beams caused by concentrated loads of joints, CIB-W18/33-7-7, Oxford, August 2000.
[7] T.A.C.M. van der Put, The tensor-polynomial failure criterion for wood polymers, www.dwsf.nl/downloads.htm, Delft, 2005.

Appendix

Derivation of the power law.

Any function $f(x)$ always can be written in a reduced variable x/x_0
$f(x) = f_1(x/x_0)$
and can be given in the power of a function:

$f(x) = f_1(x/x_0) = [\{f_1(x/x_0)\}^{1/n}]^n$ and expanded into the row:

$$f(x) = f(x_0) + \frac{x - x_0}{1!} \cdot f'(x_0) + \frac{(x - x_0)^2}{2!} \cdot f''(x_0) + ...$$

giving:

$$f(x) = \left[\{f_1(1)\}^{1/n} + \frac{x - x_0}{x_0} \frac{1}{n} \{f_1(1)\}^{1/n-1} \cdot f'(1) + ... \right]^n = f_1(1) \cdot \left(\frac{x}{x_0} \right)^n$$

when: $(f_1(1))^{1/n} = (f_1(1))^{1/n-1} \cdot f_1'(1)/n$ or: $n = f_1'(1)/f_1(1)$

where: $f_1'(1) = [\partial f_1(x/x_0)/\partial(x/x_0)]$ for $x = x_0$ and $f_1(1) = f(x_0)$

$$\text{Thus: } f(x) = f(x_0) \cdot \left(\frac{x}{x_0} \right)^n \quad \text{with} \quad n = \frac{f_1'(1)}{f_1(1)} = \frac{f'(x_0)}{f(x_0)} \qquad \text{(B.1)}$$

It is seen from this derivation of the power law, Eq.(B.1), using only the first two expanded terms, that the equation only applies in a limited range of x around x_0.